Hydrology and Earth System Sciences, 4(1), *173–183* (2000) © EGS

Hydrology & Earth System Sciences

A conditional simulation model of intermittent rain fields

Luca G. Lanza

University of Genova, Department of Environmental Engineering, 1 Montallegro, 16145 Genova, Italy
e-mail for corresponding author: luca@diam.unige.it

Abstract

The synthetic generation of random fields with specified probability distribution, correlation structure and probability of no-rain areas is used as the basis for the formulation of a stochastic space-time rainfall model conditional on rain gauge observations. A new procedure for conditioning while preserving intermittence is developed to provide constraints to Monte Carlo realisations of possible rainfall scenarios. The method addresses the properties of the convolution operator involved in generating random field realisations and is actually independent of the numerical algorithm used for unconditional simulation. It requires only the solution of a linear system of algebraic equations the order of which is given by the number of the conditioning nodes. Applications of the methodology are expected in rainfall field reconstruction from sparse rain gauge data and in rainfall downscaling from the large scale information that may be provided by remote sensing devices or numerical weather prediction models.

Keywords: Space-time rainfall; Conditioning; Stochastic models

Introduction

Although investigated by the hydrological and meteorological communities for about two decades, the problem of developing a suitable mathematical representation of the space-time variability of rainfall over a variety of scales of technical interest is still a challenge. Nearly ten years ago, Cho and Chan (1987) argued that, due to the complexity of the involved physical processes and to the lack of knowledge about many of the basic phenomena, the numerous attempts to build precipitation models able to provide reliable predictions had limited success. With respect to many of the scales of interest in the usual hydrological applications, the statement is still valid today, even if advances have been experienced in the understanding of the statistical characteristics of precipitation (Waymire and Gupta, 1981a,b,c; Zawadski, 1987; Gupta and Waymire, 1987) and on the coupling of the evidence of a hierarchical organisation of different structures at different scales – which is typical of the meteorologists' approach after the Orlanski classification (Orlanski, 1975) – with the self-similarity features of the rainfall field emerging from recent investigations (Kumar and Foufoula-Georgiou, 1993a,b; Lovejoy and Mandelbrot, 1985; Waymire, 1985).

The work presented in this paper originates from the study of extreme rainfall events which may produce flash floods and disastrous inundation in the small to medium size catchments along the highly urbanised coastal areas of the northern Mediterranean region (Lanza and Siccardi, 1994, 1995). In this case, comparative analyses between the typical morphological structures of natural drainage systems and the space-time variability of the precipitation field lead to the hypothesis that flash floods in areas of complex orography may be associated with some kind of 'resonance' between the scales of the drainage network organisation and those of the covariance and intermittence structure of precipitation. The term 'resonance' is used here to define the probability that a given space/time component of the rainfall field – that is critical for basins of a given scale – will hit the corresponding geomorphologic component of the drainage network within the target region, which is assumed as a condition for the occurrence of floods. Therefore, the resonance must be interpreted as a key to the understanding of the relation between the aggregation scales – in space and time – of the internal variability of the rainfall field, and the aggregation scale of the runoff processes at basin scale.

This 'resonance' of scales is not predictable in a deterministic sense due to the fact that, though landscape morphology is well resolved and consequently the organisation of drainage networks is known in fine detail, the inherent variability of the precipitation field is still unresolved but for a range of space-time scales a few orders of magnitude larger than those requested by the study of small to medium size basins. At finer scales, the stochastic approach is the most appropriate, because it allows quantification and control of the uncertainties that any physical model would inevitably introduce. The basic idea is that of generating, directly, Monte Carlo realisations of random fields, which preserve specified conditions imposed on their statistical structure and distribution of no-rain areas.

The contribution of the present paper is the development of a new technique allowing conditional generation of the rainfall field by constraining the field to be consistent with the rainfall rates measured at existing rain gauge locations and preserving the desired distribution of no-rain areas. The method simply addresses the properties of the convolution algorithm involved in the synthesis of random fields. Being independent of the performance of the numerical generator used for unconditional simulation, the method occupies very little CPU time, i.e. that requested by solving a linear system of algebraic equations the order of which is given by the number of the conditioning nodes. Most of the research effort aimed at the development of random field numerical generators, able to preserve the second order characteristics of the observed data-sets, was fuelled by the need to simulate two- and three-dimensional spatial realisations of log-transmissivity fields in natural aquifers, as expressed by the study of contaminant transport in heterogeneous porous media (e.g. Bellin *et al.*, 1992; Delhomme, 1979). The use of the same code for the stochastic simulation of the space-time structure of precipitation fields was not addressed to the same extent, due to the fact that the second-order description of the rainfall process is unlikely to be sufficient to explain many features of observed precipitation. However, performance comparisons with even complex different models, either of a conceptual (Waymire *et al.*, 1984) or fractal nature (Lovejoy and Mandelbrot, 1985), indicate some operational advantages which make direct simulation models very promising for usual hydrological applications: the number of parameters they require is very limited and they show extraordinary reproduction capabilities in the face of radar rainfall maps.

Stochastic models of space-time rainfall have indeed been addressed in hydrological applications such as multiple-sensor network design problems (Krajewski, 1987; Azimi-Zoonoz *et al.*, 1989; Seo *et al.*, 1990; Krajewski *et al.*, 1993), the analysis of rainfall input accuracy effects on the performances of hydrological models (Krajewski *et al.*, 1991; Wilson *et al.*, 1979), the study of satellite derived rainfall estimates (Bell, 1987; Bell *et al.*, 1990), etc.

The organisation of the paper is as follows. The space-time model of rainfall used in this paper is dealt with first. After a definition of the conceptual framework for this model, its mathematical structure, due to Bell (1987), is recalled briefly. Examples of two-dimensional realisations of the rainfall field are presented, which preserve specified second-order statistics, correlation structure and the distribution of no-rain areas. The conditional generation technique is described in the case of one-dimensional simulations and single-site conditioning. The extension to higher dimensions and to multiple-site conditioning is a question of simple algebraic developments and only results are presented. In the conclusions, possible applications of the proposed methodology in different hydrological studies are discussed and further developments and research directions are identified.

A model for space-time rainfall

Various stochastic models of space-time rainfall have been presented in the literature, initially led by the extension of temporal rainfall models to the two-dimensional case (Waymire *et al.*, 1984). The implementation of a modified Turning Band algorithm (see later in this section) was recently presented by Mellor (1996), Mellor and O'Connell (1996) and Mellor and Metcalfe (1996). Other examples may be found in the work of Smith and Karr (1985), Krajewski and Georgakakos (1985), Bell (1987), Smith and Krajewski (1987), Rodriguez-Iturbe and Eagleson (1987), Lebel *et al.* (1987) and in the review paper by Creutin and Obled (1982).

The present work takes its cue from the approach of Bell (1987) who developed a random field generation model which preserves the main statistical and structural characteristics of the observed rainfall field: a brief description of this model is presented later. The basic hypothesis is that of considering the rainfall field as suitably represented by means of a homogeneous regionalised variable:

$$R_{A,\Delta T}(x_A, y_A) = \frac{1}{A}\int_A \frac{1}{\Delta T}\int_{\Delta T} r(x,y)\, dA\, dt \tag{1}$$

where $r(x,y)$ is the instantaneous rainfall intensity at any point with co-ordinates (x,y), A is the support area and ΔT the time step of integration.

The model of space-time rainfall presented in this paper assumes that the rainfall process is statistically homogeneous, or stationary. Statistical homogeneity is the physical counterpart of stationarity and implies that statistical properties of the underlying random process depend only on the separation distance and not on the actual location of the grid nodes.

The random variable under examination presents a positive probability (say $1 - p$) to be zero while it is continuous – i.e. such that $\Pr[R = r] = 0$ – elsewhere. This is a mixed distribution which can be described through its probability density function (Kedem *et al.*, 1990):

$$h(r) = 0 \qquad r < 0 \tag{2a}$$

$$h(r) = 1 - p \quad r = 0 \tag{2b}$$

$$h(r) = p\ f(r) \quad r > 0 \tag{2c}$$

where f(r) is the probability density function of R conditional on $R > 0$.

Though a few experimental studies on the statistical characteristics of observed rainfall fields have been presented in the literature, using radar data at different resolution scales, there is no general agreement upon the nature of f(r). The earlier studies of Neymann and Scott (1967) suggest the use of a gamma distribution. Lovejoy and Mandelbrot (1985) argue that the hyperbolic distribution is well suited to represent the rainfall process and use that observation to support the fractal hypothesis of the nature of space-time precipitation. The lognormal distribution was suggested by Houze and Cheng (1977) and Kedem *et al.* (1990) using experimental data and by Kedem and Chiu (1987) using a theoretical approach which, based on a few assumptions on the rainfall process, demonstrates that the spatial process is lognormally distributed.

When f belongs to a family of parametric distributions with parameter vector θ, the notation f(r) is replaced by f(r, θ) and h(r) by h(r, p, θ). Let $\theta = (\mu, \sigma^2)$, and

$$f(r,\theta) = \frac{1}{r\cdot\sigma\cdot\sqrt{2\pi}}\cdot\exp\left[-\frac{1}{2\sigma^2}\cdot(\ln r-\mu)^2\right] \qquad r>0$$

$$f(r,\theta) = 0 \qquad r\le 0 \tag{3}$$

In this case R has a lognormal mixed distribution with parameters p, μ, σ^2, and we have (Kedem *et al.*, 1990):

$$\begin{aligned} E[R] &= p\cdot\exp(\mu+\sigma^2/2) \\ VAR[R] &= p\cdot\exp(2\mu+\sigma^2)[\exp(\sigma^2)-1] \end{aligned} \tag{4}$$

Several techniques for the synthesis of random fields with specified covariance structure have been proposed in the literature, based on various numerical algorithms. A detailed performance comparison (Bellin, 1991) of three of the most used numerical methods – namely the Turning Band Method (Mantoglou and Wilson, 1982; Tompson *et al.*, 1989; Dietrich, 1995, 1996; Gneiting, 1996), the Matrix Decomposition Method (Davis, 1987; Fai Ma and Mills, 1987) and the Direct Fourier Transform (DFT) method (Gutjahr, 1989) – leads to the conclusion that the spectral method based on DFT provides the best results in terms of accuracy and computational efficiency.

The synthetic generation of a Gaussian random field with zero mean and unit variance is addressed here as the basis for space-time rainfall simulation. The Direct Fourier Transform method presented by Gutjahr (1989) is used. The algorithm generates random fields of real variables on a regular grid by directly performing an inverse Fourier transform on the randomised discrete spectral representation of the variable itself. The method uses the theorem of spectral representation applied to a statistically homogeneous random field of spectral density $S(k)$ which states that, if $V(x)$ is a statistically homogeneous random field with zero mean and spectral density $S(k)$, then a unique complex stochastic process $Z(k)$ exists such that:

$$V(x) = \int_{-\infty}^{\infty}\exp(ix\cdot k)dZ(k) \tag{5a}$$

$$E[dZ(k)] = 0 \qquad \forall k \tag{5b}$$

$$E[dZ(k_1)dZ^*(k_2)] = \begin{cases} 0 & \text{if} \quad k_1\neq k_2 \\ S(k)dk & \text{if} \quad k_1=k_2=k \end{cases} \tag{5c}$$

where in Eqn. (5a), $i=(-1)^{1/2}$, k is the wave number or the angular frequency vector, and $dZ(k)$ is a complex variable of the form:

$$dZ = dZ_R + i\cdot dZ_I \tag{6}$$

where the subscripts R and I denote the real and imaginary components, respectively. This theorem allows the representation of a correlated structure in the spatial domain, $V(x)$, through a non correlated structure, $Z(k)$, in the frequency domain. The approach is consonant with the Fourier transform approach given later in this paper. In order to perform the inverse Fourier transform it is convenient – following Robin *et al.* (1993) – to switch from the angular frequency k to the spatial frequency u according to:

$$k = 2\pi\cdot u \tag{7}$$

to obtain

$$\Phi(u) = (2\pi)^q\cdot S(k) \tag{8}$$

and

$$\hat{V}(x) = \int_{-\infty}^{\infty}\exp(i\cdot 2\pi u\cdot x)d\hat{Z}(u) \tag{9}$$

where $\Phi(u)$ is the power spectral density function, q is the number of dimensions and the units of u are cycles per unit distance.

Equation (9) has the form of a continuous inverse Fourier transform, which can be evaluated numerically with the following approximations: (a) the frequency domain is discretised into a finite number of frequency intervals Δu in each dimension, and (b) the frequency domain must be truncated at a finite frequency, known as the Nyquist frequency, at each end of the domain. By writing down the representation integral as a discretised and truncated sum the construction of $\hat{V}$ involves the generation of the independent random variables dZ_R and dZ_I – with zero mean and variance 1/2 – only. Assume now that a synthetic Gaussian random field $G(x)$, with zero mean and unit variance, is generated with some specified spatial correlation structure

$$c_G(h) = c_G(|x-y|) = E[g(x)g(y)] \tag{10}$$

where $E[\cdot]$ indicates the expected value operator and $g(x)$ is the realisation of $G(x)$ at location x.

Following Bell (1987), the random field $g(x)$ is transformed into a rainfall field $r(x)$ through a suitable transfer function $\mathfrak{R}$:r = $\mathfrak{R}$(g) so as to present a mixed probability distribution, i.e. accounting for a specified percentage (1 − p) of null values and being distributed as f(r) elsewhere. To this aim a threshold value, g_0, is defined such that the Gaussian field exceeds g_0 for a percentage p of space. The portion of the field that exceeds the threshold is then re-scaled so that its distribution is precisely f(r). In other words, the variable g is converted into a variable u uniformly distributed in the interval [0,1]:

$$u = G(g) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{g} \exp\left(-\frac{1}{2}\xi^2\right) d\xi \qquad (11)$$

To have $r = 0$ for $u < (1 - p)$, the threshold g_0 will be such that:

$$G(g_0) = (1 - p) \qquad (12)$$

and the values of u greater than (1 − p) are scaled through the complement of the cumulative distribution of f(r):

$$C(r) = \int_r^{\infty} f(s)\, ds \qquad \textit{setting} \qquad \begin{cases} r = C^{-1}[(1-u)/p] \\ u > (1-p) \end{cases} \qquad (13)$$

where C^{-1} is the inverse functional of C. The transform $\mathfrak{R}$ is thus defined as follows:

$$\begin{aligned} \mathfrak{R}(g) &= 0 & g \le g_0 \\ \mathfrak{R}(g) &= C^{-1}[(1 - G(g))/p] & g > g_0 \end{aligned} \qquad (14)$$

The effect of the transformation $\mathfrak{R}$ on the correlation of the Gaussian field $c_G(h)$ has been investigated by Bell (1987) and the results are briefly summarised here. The corresponding correlation generated by the transformation $\mathfrak{R}$ is:

$$c_r(h) = \gamma(c_G(h)) \qquad (15)$$

Analytical expressions for $\gamma(c_G)$ may be obtained for special cases of the transformation $\mathfrak{R}$. In particular, when the Gaussian variable G is transformed into a lognormal variable via $r = exp(\mu + \sigma g)$ we have (Mejia and Rodriguez-Iturbe, 1974):

$$c_r = \gamma(c_G) = \frac{\exp(c_G \cdot \sigma^2) - 1}{\exp(\sigma^2 - 1)} \qquad (16)$$

Given γ, the proper correlation function to be imposed on the Gaussian field in order to obtain the desired correlation of the rainfall field is then:

$$c_G(h) = \gamma^{-1}(c_r(h)) = \frac{1}{\sigma^2} \ln[1 + (\exp \sigma^2 - 1) \cdot c_r(h)] \qquad (17)$$

where $c_r(h)$ is the desired correlation function. As an example, to obtain a lognormal field with correlation:

$$\begin{aligned} &\text{exponential:} & c_r(h) &= \exp(-\alpha \cdot h) \\ &\text{or Whittle (Bessel):} & c_r(h) &= \alpha \cdot h \cdot K_1(\alpha\xi) \end{aligned} \qquad (18)$$

with K_1 representing the modified Bessel function of the first type and order one and α being any positive parameter, the correlation functions to be imposed on the Gaussian field will be, respectively:

$$\begin{aligned} c_G(h) &= \frac{1}{\sigma^2} \ln[1 + (\exp(\sigma^2) - 1) \exp(-\alpha \cdot h)] \\ c_G(h) &= \frac{1}{\sigma^2} \ln[1 + (\exp(\sigma^2) - 1) \cdot \alpha h \cdot K_1(\alpha\xi)] \end{aligned} \qquad (19)$$

The spectral density functions will be, in the two cases:

$$\begin{aligned} S(k) &= \frac{2}{\pi\sigma^2} \int_0^{\infty} \ln[1 + (b-1) \exp(-\alpha \cdot \xi)] \cos(k\xi) d\xi \\ S(k) &= \frac{2}{\pi\sigma^2} \int_0^{\infty} \ln[1 + (b-1) \cdot \alpha\xi \cdot K_1(\alpha\xi)] \cos(k\xi) d\xi \end{aligned} \qquad (20)$$

where $b = exp(\sigma^2)$, and the integral scales become:

$$\begin{aligned} \lambda &= \frac{1}{\sigma^2} \int_0^{\infty} \ln[1 + (b-1) \exp(-\alpha \cdot \xi)] d\xi \\ &= \frac{1}{\alpha\sigma^2} \int_0^{b-1} \frac{\ln(1+y)}{y} dy \end{aligned} \qquad (21)$$

with $y = (b-1)exp(-\alpha\xi)$ and:

$$\lambda = \frac{1}{\sigma^2} \int_0^{\infty} \ln[1 + (b-1) \cdot \alpha\xi \cdot K_1(\alpha \cdot \xi)] d\xi \qquad (22)$$

respectively. Thus $\lambda = S(k)\, \pi/2$; note that λ is a function of α and σ^2.

In the exponential case, numerical solutions of the integral in Eqn (21) can be obtained to get the relationship $\alpha = \alpha(\lambda)$ for a given value of σ^2, and the spectrum for any k may be also obtained by means of numerical integration or by Fourier Transform. Four examples of synthetically generated rainfall fields with an exponential correlation function are presented in Fig. 1 where two-dimensional realisations over a grid of 256 × 256 nodes are depicted. The underlying Gaussian field has the same statistical parameters in all realisations while the amplitude of no-rain areas varies from 1-p = 0.8 to 1-p = 0.2; the same initial Gaussian realisation has been used in the four rainfall fields so as to show the effect of the transformation $\mathfrak{R}$ with various no-rain area thresholds.

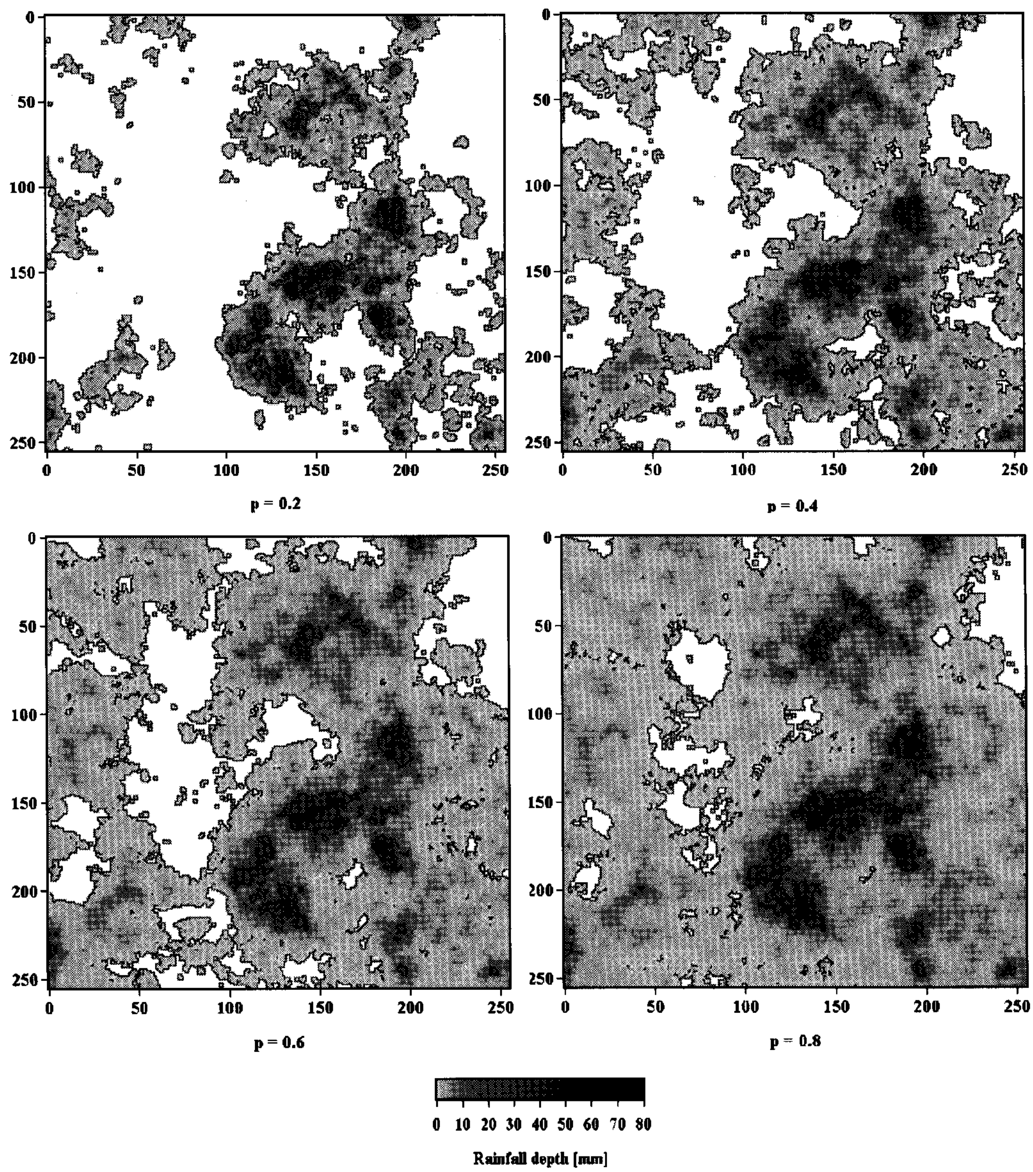

Fig. 1. *Synthetic realisations of the two-dimensional rainfall field generated over a grid of 256 × 256 nodes. The rainfall field has the same statistical parameters in all realisations while the intermittence varies from 1-p = 0.8 (upper left) to 1-p = 0.2 (lower right). The same unconditional Gaussian realisation has been used in the four cases.*

The conditional generator

The synthetic generation of random fields with specified covariance structure and probability of no-rain areas described earlier is able to reproduce the observed rainfall field only in a statistical sense. Each of the generated rainfall maps, either two- or three-dimensional, is just one of the possible rainfall scenarios with the specified characteristics. Reducing the ensemble of the possible realisations by constraining the generated random field is necessary to reproduce further known characteristics of the rainfall field. The simplest case is that of conditional generation, where a fixed number of grid points is assigned some measured rainfall value (i.e. rainfall depth in a given time step) as provided by any available rain gauge station. A further step is that of conditioning on area averaged values that may be obtained by sampling the observed rainfall field through a given support area, as is the case of satellite derived observations at a larger scale or even the output of numerical weather prediction models.

In this paper, the analysis will be limited to conditional generation of rainfall fields based on grid-point values, with reference to rainfall depths as observed in a fixed number of rain gauges. A new method for conditional simulation based on grid-point values is presented: the idea is to adjust the value of the unconditionally simulated field to match the value of the conditioning nodes while preserving the desired correlation structure and intermittence. The method is based on the properties of the Fourier transforms in the face of the convolution operation, i.e. the mathematical tool actually exploited by random field generation algorithms based on DFT. For the sake of conciseness, the description of the method is addressed in the following with reference to the one-dimensional case: the extension to the two- and three-dimensional cases is, however, straightforward.

The basic concepts and mathematical properties of the Fourier Transform, and the representation of the convolution and correlation algorithms can be expressed through the combination of two functions – say in the time domain for simplicity – $z(t)$ and $\omega(t)$, and the respective Fourier transforms $Z(f)$ and $\Omega(f)$ in the frequency domain.

The convolution theorem states:

$$z * \omega \Leftrightarrow Z(f) \cdot \Omega(f) \tag{23}$$

where '*' is the convolution operator and the notation '⇔' indicates a Fourier transforms pair. In other words, the Fourier transform of the convolution of two functions is the product of the Fourier transforms of the two functions:

$$\mathrm{F}(z * \omega) = \mathrm{F}(z) \cdot \mathrm{F}(\omega) \tag{24}$$

where F indicates the Fourier transform operator.

The main property of interest, as regards the operator recalled above, is linearity. The most important consequence of this property for application in the present context is that the sum of independent solutions of the convolution problem is still a solution of the same problem.

This means that a random field with given covariance structure as generated through some spectral method may be obtained alternatively by adding two independent solutions of the convolution problem. By choosing one of them as the specific solution obtained by convolution of an impulse of proper amplitude, and the other one as a generic (thus independent) solution, the resulting field is still a solution of the generation problem where the value at the impulse location may be specified as desired.

The specified value at any conditioning node is preserved exactly and the randomness and correlation structure of the neighbouring region are preserved as well. Also, the conditioning nodes are not forced to act as local maxima, as is the usual inconvenience of many of the available conditional simulation techniques.

Note that the convolution of an impulse of unit amplitude with the correlation function produces the shape of the correlation function peaked at the location of the impulse itself. The eventual transformation of the random field from the log-space into the natural space, to obtain a rainfall field with given probability distribution of no-rain areas as described in the previous section, does not affect this methodology except in the determination of the proper amplitude to be set for the impulse function. An analytical expression for the amplitude of the conditioning impulse can be derived using for numerical convenience

$$u = 1 - G(g) = \frac{1}{2} erfc\left(\frac{g}{\sqrt{2}}\right) \tag{25}$$

instead of Eqn. (11) to generate a uniformly distributed variable (Bell, 1987). The effect of the transformation $r = \mathfrak{R}(g)$ at any single node can thus be written, using Eqns. (13) and (14), as:

$$r = \exp(\mu + \sigma\xi) \tag{26}$$

where μ and σ^2 are the mean and variance expected from the generated field, and

$$\xi = \sqrt{2}\, erf^{-1}\left(1 - \frac{2[1 - G(g)]}{p}\right) \tag{27}$$

The error function $erf(\cdot)$ is defined as:

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{28}$$

and the complementary error function is simply:

$$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt \tag{29}$$

The following expression is thus obtained for $r = \mathfrak{R}(g)$:

$$r = \exp\left\{\mu + \sigma\sqrt{2}\, erf^{-1}\left[1 - \frac{1}{p}\left[1 - erf\left(\frac{g}{\sqrt{2}}\right)\right]\right]\right\} \tag{30}$$

Equation (30) can be written in terms of any conditioning value $r^* = r(x^*)$ and inverted in order to obtain the corresponding amplitude of the function $g^* = g_N(x^*)$, as required in the conditional field $g_N(x)$, in the form:

$$g^* = \sqrt{2}\, erf^{-1}\left\{1 - p + p \cdot erf\left[\frac{\ln(r^*) - \mu}{\sigma\sqrt{2}}\right]\right\} \tag{31}$$

The value of the unconditional simulation at x^* is $g_V^* = g_V(x^*)$, so the shift to be added to the unconditional field at x^* is $g^* - g_V^*$.

The field $g_N(x)$ obtained by adding the convolution of such an impulse to the unconditional realisation, will give the desired value at x^* and is thus given by:

$$g_N(x) = g_V(x) + [\tilde{g}^* - g_V^*] \cdot c_G(|x - x^*|) \tag{32}$$

where:

$$\tilde{g}^* = \frac{g^* - \mu_N}{\sigma_N} \tag{33}$$

in which μ_N and $\sigma^2{}_N$ are the mean and variance of the final field.

Here $g_V(x)$ is the unconditional Gaussian field, thus a generic solution of the convolution problem, g^*_V is the value of the unconditional field at the conditioning node x^*, and $c_G(|x - x^*|)$ is the amplitude of the correlation function at a distance $|x - x^*|$. At x^* the conditional field g_N degenerates to $\tilde{g}_N(x^*) = \tilde{g}^*$ because $c_G(0) = 1$. The conditioning rule at a single site is illustrated in Fig. 2 for the case of one-dimensional fields.

For the sake of simplicity, the possible numerical effects of the involved operations on the statistics of the generated field were not mentioned in the above developments. Actually, the sum of the unconditional field N[0,1] with the convolution of an impulse of given amplitude produces an output field where the correlation structure is preserved but the zero mean and unit variance characteristics are not. Note that p need not be preserved at this step because the transformation producing intermittence in the field is made at a later time and the addition above is between two fields where the probability of null values equals zero. In the theoretical case of infinite space-time domain, this effect would be negligible. In practice, however, such effect needs to be taken into account by normalising the combined field $g_N(x)$ while preserving g^* as derived from Eqn. (31); to account for the normalisation process, the amplitude of the impulse function at the conditioning node must be modified according to Eqn. (33).

In case the field is sufficiently large, μ_N and σ^2_N are given by:

$$\begin{aligned} \mu_N &= \mu_V + \tilde{g}^* \cdot \mu_c = \tilde{g}^* \cdot \mu_c \\ \sigma_N^2 &= \sigma_V^2 + \tilde{g}^{*2} \cdot \sigma_c^2 - \sigma_c{}^2 = 1 + \sigma_c^2 [\tilde{g}^{*2} - 1] \end{aligned} \tag{34}$$

where $\mu_V = 0$ and $\sigma^2_V = 1$ are the statistical parameters of the unconditional field, μ_c and σ^2_c are the mean and variance of the field obtained as a convolution of the impulse function, and $\tilde{g}^*$ is the amplitude of the impulse function.

The values for μ_N and σ^2_N can be obtained by eliminating $\tilde{g}^*$ between Eqns. (33) and (34), yielding:

$$\mu_N = \left[\frac{g^* - \mu_N}{\sigma_N}\right] \cdot \mu_c \tag{35a}$$

$$\sigma_N = 1 - \sigma_c{}^2 + \left[\frac{g^* - \mu_N}{\sigma_N}\right]^2 \cdot \sigma_c{}^2 \tag{35b}$$

Extension of the overall methodology to the case of n conditioning nodes requires the convolution influence of the impulse amplitudes on each other to be considered. The problem is reduced to the solution of an algebraic system of equations in the form:

$$[G] \cdot [\alpha] = [\tilde{g}^*] \tag{36}$$

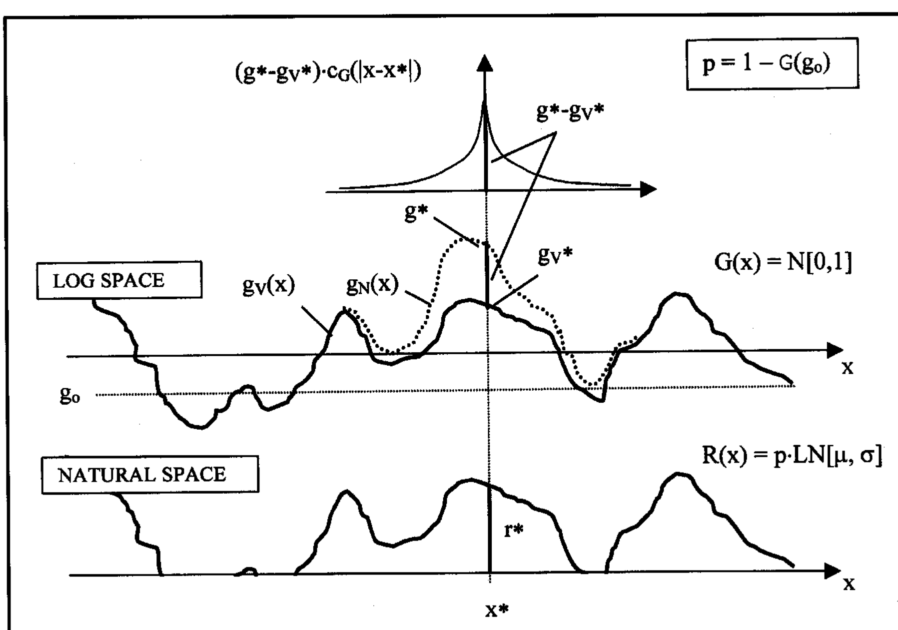

Fig. 2. *Indicative sketch of the one-dimensional conditioning rule at a single point x* where a value r* for the rain field is desired. The convoluted impulse with amplitude (g* – gv*) is added to the unconditional random field G(x) in the log space to obtain a new field g_N which, after transformation through $\Re$ will yield the conditional rain field R(x).*

where

$$[G] = \begin{pmatrix} 1 & c_G(d_{1,2}) & \dots & c_G(d_{1,n}) \\ c_G(d_{2,1}) & 1 & \dots & c_G(d_{2,n}) \\ \dots & \dots & \dots & \dots \\ c_G(d_{n,1}) & c_G(d_{n,2}) & \dots & 1 \end{pmatrix}$$

with $c_G(d_{i,j}) = c_G(|x_i - x_j|)$ and $c_G(d_{i,j}) = c_G(d_{j,i})$;

$[\tilde{g}^*]$ is the vector of the amplitudes of the impulse function in the conditioning nodes; $[\alpha]$ is the unknowns vector containing the weighted contributions of the conditioning nodes.

The expression of μ_N and σ_N in this latter case is:

$$\mu_N = \mu_c \cdot \sum_{i=1}^{n} \alpha_i \tag{37a}$$

$$\sigma^2 N = 1 + \sigma_c^2 \sum_{i,j=1}^{n} (\alpha_i \alpha_j - \beta_i \beta_j) \tag{37b}$$

where $[\beta] = \beta_k$ ($k = 1, \dots n$) is the solution of the system

$$[G] \cdot [\beta] = [U] \tag{38}$$

$[U]$ being a unit vector.

The last term in Eqn (37b) accounts for variance reduction due to the correlation of the conditioning function with the original field.

In Fig. 3 the procedure for multiple-site conditioning is illustrated in the two-dimensional case. The steps of the procedure are represented, starting from (a) the initial Gaussian field realisation, with zero mean, unit variance and exponential de-correlation and (b) the unconditioned rainfall field with percentage of rain areas 1-p = 0.2. In the central picture (c) the effect of the convolution of five different impulses with specified amplitudes at five conditioning nodes is represented in the log-space. This is combined with the unconditional Gaussian field (a) in order to obtain a random field (d) which will produce – after transformation through $\mathfrak{R}$ – the desired intermittent rainfall field (e) duly conditioned on multiple-site observations.

Note that the method used for interpolation of the impulse functions out of the conditioning nodes is close to conventional kriging (Matheron, 1971), though it uses covariance as a basis function. The objective, however, is not that of searching here for any optimal interpolator characteristics as the only requirement is that the basis function must be a solution of the convolution problem. Equation (36) provide a simple way to achieve such a condition, though any other interpolation could be used, provided it satisfies the requirements above.

The core of the methodology is Eqn. (31), together with the system in Eqns. (35a,b), which ensure that – after transformation through $\mathfrak{R}$ – the resulting rain field, conditional on rain gauge observations, will preserve exactly the imposed second order statistics and the specified intermittence as well.

Conclusions

A methodology has been presented to perform conditional simulation of rain fields, where intermittence – intended as the probability of no-rain areas – is preserved exactly while conditioning is solved analytically. The operation is performed in the log-space and conditioning parameters are evaluated so as to reproduce the desired figures once the field is transformed through the functional R into the natural space.

The relevance of modelling intermittence in practical applications is strongly dependent on the space and time scales involved in the representation of the rain field. The distribution of no-rain areas may play a significant role when simulation of the rain process is required at relatively fine scales. This is the case, for instance, in urban hydrology and the management of small to medium size catchments (10 – 100 km^2), especially when the interest is on extreme events and prediction of critical runoff conditions is required. Another example is the management of wastewater treatment plants in urban environments for control of the highly polluted initial rain waters that need to be collected and routed to the plant.

In general terms, conditional simulation of a rain field that includes intermittence as a parameter is useful when the available monitoring system fails to cover a significant range of fine space-time scales and thus to reproduce the process with accuracy. The typical case is that of a coarsely spaced network of rain gauges that are, on average, more distant to each other (for a given event or class of events) than the characteristic spatial scale of the intermittence process. Simulation of the rain field with the proposed methodology allows one to take care, at least in a statistical sense, of the role of intermittence in determining the space-time variability of precipitation.

The distribution of no-rain areas is often neglected in conditional simulation problems because of the difficulty of estimating p in quantitative terms based on the available monitoring devices. Radar maps are usually obtained at a sufficiently fine resolution for assessment of the distribution of no-rain areas, assuming that a physical lower scale limit exists for intermittence around a few hundred metres. However, they represent an indirect measure of precipitation which also involves some modelling of the atmospheric column at each location, and the accuracy of the rain maps has not been fully demonstrated yet, especially in the case of heavy rainfall.

Where the radar is not available, rain gauges can be used to derive such information. Provided coherence is preserved between the space and time scales of observation, intermittence in space can be derived from that observed in time and used as an input parameter for the algorithm

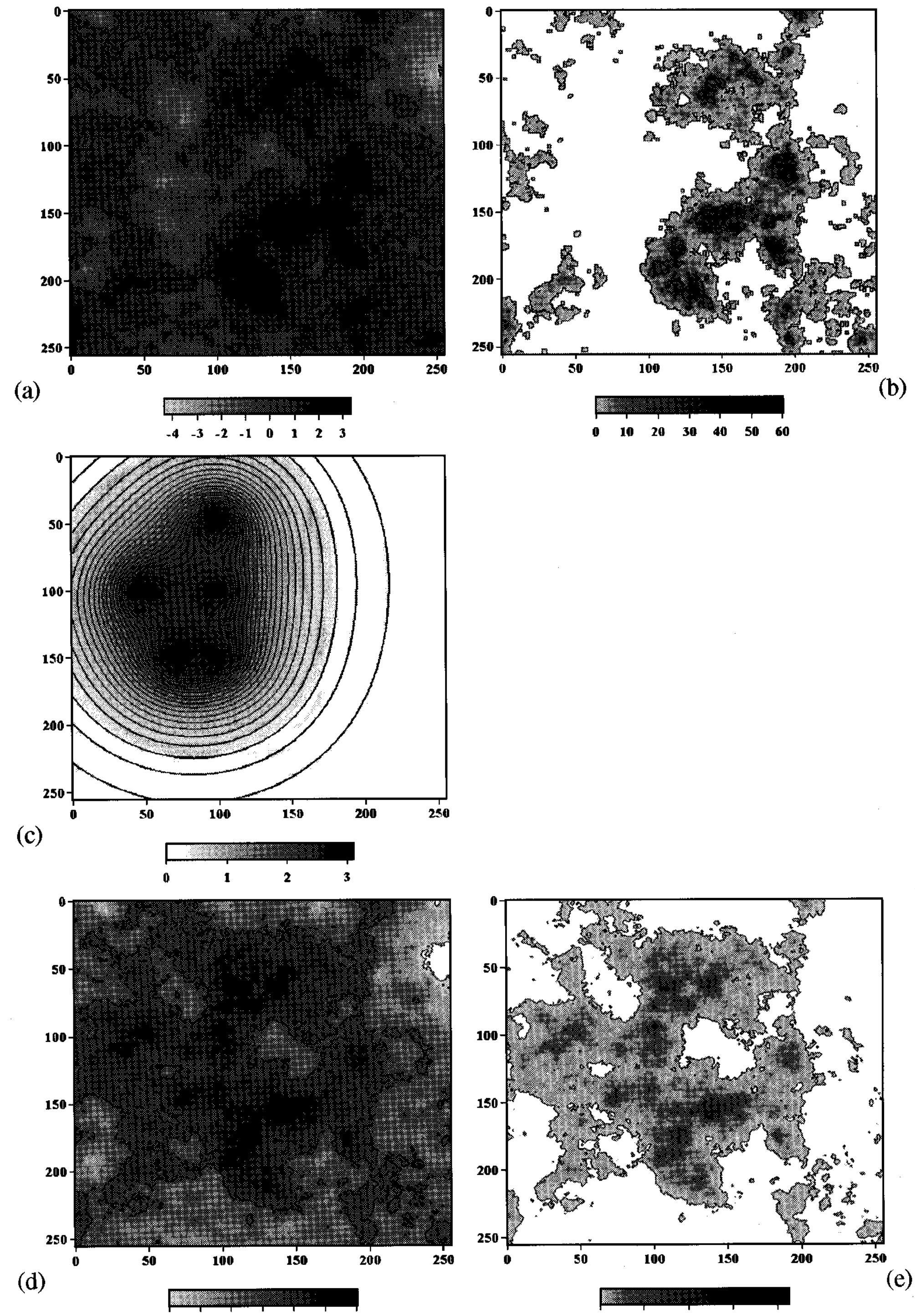

Fig. 3. *The subsequent steps of conditional simulation in two dimensions: (a) the unconditional Gaussian field with zero mean, unit variance and exponential decorrelation (upper left); (b) the obtained rainfall field with intermittence 1-p = 0.8 (upper right); (c) the convoluted effects of five different impulses with specified amplitudes at five conditioning nodes (central picture); (d) the modified Gaussian field obtained as a sum of (a) and (c) (lower left); and (e) the desired intermittent rainfall field (lower right) duly conditioned on multiple-site observations.*

proposed. One implication of the methodology, however, is that the resulting structure of intermittence in space is controlled by the underlying unconditional field, i.e. is the same in both the spatial and temporal domain and does not differ from that of the rain process. Should observations not validate this assumption, the model fails to reproduce such variability.

Conditional simulation was addressed in this work with reference to some point measurement obtained at the available rain gauge sites. The next step is to allow for conditioning on area averaged values, which can be derived from observations at some larger scales than those used for simulation of the rain field. These can be obtained, for instance, from remote sensing using radiometers borne on geostationary and/or sun-synchronous satellite platforms (Smith *et al.*, 1992; Mugnai *et al.*, 1993). In the near future, the output of physically based models of the atmosphere will probably achieve the resolution and accuracy requirements to be used as a surrogate of such measurements. In this case, the methodology presented could be used as a downscaling algorithm, conditional on both point and area averaged constraints.

Acknowledgements

This work was developed by the author as partial fulfilment of his PhD thesis at the University of Genova. Thanks are due to Profs. Franco Siccardi and Paolo La Barbera, also from the University of Genova, for their valuable advice and contribution to this research. The comments of Profs. Ezio Todini and Geoffrey Pegram allowed substantial improvement of this work and are therefore gratefully acknowledged. The research was partially supported by the Ministry of Scientific Research and University (MURST) under the national project "Climatic and antropic effects on hydrological processes".

References

Azimi-Zoonoz, A., Krajewski, W.F., Bowes, D.S. and Seo, D.J., 1989. Spatial rainfall estimation by linear and non-linear co-kriging of radar rainfall and raingage data, *Stoch Hydrol. Hydraul.*, **3**, 51–67.

Bell, T.L., 1987. A space-time stochastic model of rainfall for satellite remote sensing studies, *J. Geophys. Res.*, **92**, 9631–9643.

Bell, T.L., Abdullah, A., Martin, R.L. and North, G.R., 1990. Sampling errors for satellite-derived tropical rainfall: Monte Carlo study using a space-time stochastic model, *J. Geophys. Res.*, **95**, 2195–2205.

Bellin, A., 1991. *Numerical generation of random fields with specified spatial correlation structure*, Int. Rep. No IDR 2/1991, University of Trento, Italy.

Bellin, A., Salandin, P. and Rinaldo, A., 1992. Simulation of dispersion in heterogeneous porous formations: statistics, first order theories, convergence of computations, *Water Resour. Res.*, **28**, 2211–2227.

Cho, H.R. and Chan, D.S.T., 1987. Mesoscale atmospheric dynamics and modeling of rainfall fields, *J. Geophys. Res.*, **92**, 9687–9692.

Creutin, J.D. and Obled, C., 1982. Objective analyses and mapping techniques for rainfall fields: an objective comparison. *Water Resour. Res.*, **18**, 413–431.

Davis, M.W., 1987. Production of conditional simulations via the LU triangular decomposition of the covariance matrix, *Math. Geol.*, **19**, 91–98.

Delhomme, J.P., 1979. Spatial variability and uncertainty in groundwater flow parameters: a geostatistical approach, *Water Resour. Res.*, **15**, 269–280.

Dietrich, C.R., 1995. A simple and efficient space domain implementation of the turning bands method, *Water Resour. Res.*, **31**, 147–156.

Dietrich, C.R., 1996. Reply to Comment on "A simple and efficient space domain implementation of the turning bands method" by T. Gneiting, *Water Resour. Res.*, **32**, 3397–3399.

Fai Ma, M.S.W. and Mills, W.H. 1987. Correlation structuring and the statistical analysis of steady-state groundwater flow, *SIAM J. Sci. Stat. Comput.*, **8**, 848–867.

Gneiting, T., 1996. Comment on "A simple and efficient space domain implementation of the turning bands method" by C.R. Dietrich, *Water Resour. Res.*, **32**, 3391–3396.

Gupta, V.K. and Waymire, E., 1987. On Taylor's hypothesis and dissipation in rainfall, *J. Geophys. Res.*, **92**, 9657–9660.

Gutjahr, A.L., 1989. *Fast Fourier Transform for random field generation*. Rep. For Los Alamos Grant to New Mexico Tech., Contract No 4-R58-2690R.

Houze, R.A. and Cheng, C.P., 1977. Radar characteristics of tropical convection observed during GATE: mean properties and trends over the summer season, *Mon. Weath. Rev.*, **105**, 964–980.

Kedem, B. and Chiu, L.S., 1987. On the lognormality of rain rates, *Proc. Nat. Acad. Sci. USA.*, **84**, 901–905.

Kedem, B., Chiu, L.S. and North, G.R., 1990. Estimation of mean rain rate: application to satellite observations, *J. Geophys. Res.*, **95**, 1965–1972.

Krajewski, W.F., 1987. Cokriging radar rainfall and rain gauge data, *J. Geophys. Res.*, **92**, 9571–9580.

Krajewski, W.F. and Georgakakos, K.P., 1985. Synthesis of radar rainfall data, *Water Resour. Res.*, **21**, 764–768.

Krajewski, W.F., Lakshmi, V., Georgakakos, K.P. and Jain, S., 1991. A Monte Carlo study of rainfall sampling effect on a distributed catchment model, *Water Resour. Res.*, **27**, 119–128.

Krajewski, W.F., Raghavan, R. and Chandrasekar, V., 1993. Physically based simulation of radar rainfall data using a space-time rainfall model, *J. Appl Meteorol.*, 32, 268–283.

Kumar, P. and Foufoula-Georgiou, E., 1993a. A multicomponent decomposition of spatial rainfall fields, 1, Segregation of large- and small-scale features using wavelet transforms, *Water Resour. Res.*, **29**, 2515–2532.

Kumar, P. and Foufoula-Georgiou, E., 1993b. A multicomponent decomposition of spatial rainfall fields, 2, Self similarity in fluctuations, *Water Resour. Res.*, **29**, 2533–2544.

Lanza, L. and Siccardi, F., 1994. Distributed warnings, in: *Advances in Distributed Hydrology*, R. Rosso, A. Peano, I. Becchi and G.A. Bemporad, Eds. pp. 293–306, Water Resources Pub., Highlands Ranch, Co.

Lanza, L. and Siccardi, F., 1995. Hydrometeorological hazard in a changing perspective, *Surv. Geophys*, **16**, 137–139.

Lebel, T., Bastin, G., Obled, C. and Creutin, J.D., 1987. On the accuracy of areal rainfall estimation: a case study, *Water Resour. Res.*, **23**, 2123–2134.

Lovejoy, S. and Mandelbrot, B.B., 1985. Fractal properties of rain and a fractal model, *Tellus*, **37A**, 209–232.

Mantoglou, A. and Wilson, J.L., 1982. The turning band method for simulation of random fields using line generation by a spectral method, *Water Resour. Res.*, **18**, 1379–1394.

Matheron, G., 1971. *The theory of regionalised variables and its applications*, Cahiers du centre de morphologie mathematique, N° **5**, Fontainebleau, France.

Mejia, J. and Rodriguez-Iturbe, I., 1974. On the synthesis of random fields sampling from the spectrum: an application to the generation of hydrologic spatial processes, *Water Resour. Res.*, **10**, 705–711.

Mellor, D., 1996. The Modified Turning Bands (MTB) model for space-time rainfall. I. Model definition and properties, *J. Hydrol.*, **175**, 113–127.

Mellor, D. and Metcalfe, A.V., 1996. The Modified Turning Bands (MTB) model for space-time rainfall. III. Estimation of the storm/rainband profile and a discussion of future model prospects, *J. Hydrol.*, **175**, 161–180.

Mellor, D. and O'Connell, P.E., 1996. The Modified Turning Bands (MTB) model for space-time rainfall. II. Estimation of raincell parameters, *J. Hydrol.*, **175**, 129–159.

Mugnai, A., Smith, E.A. and Tripoli, G.J., 1993. Foundations for statistic-physical precipitation retrieval from passive microwave satellite measurements, Part II: Emission source and generalised weighting function properties of a time-dependent cloud radiation model, *J. Appl. Meteorol.*, **32**, 17–39.

Neyman, J. and Scott, E.L., 1967. Some outstanding problems relating to rain modification, *Proc. Berkley Symp. Math. Stat. Probl.*, **5**, 293–326.

Orlanski, I., 1975. A rational subdivision of scales for atmospheric processes, *Bull. Amer. Meteorol. Soc.*, **6**, 527–530.

Robin, M.J.L., Gutjahr, A.L., Sudicky, E.A. and Wilson, J.L., 1993. Cross-correlated random field generation with the direct Fourier transform method, *Water Resour. Res.*, **29**, 2385–2397.

Rodriguez-Iturbe, I. and Eagleson, P.S., 1987. Mathematical models of rainstorm events in space and time, *Water Resour. Res.*, **23**, 181–190.

Seo, D.J., Krajewski, W.F. and Bowes, D.S., 1990. Statistic interpolation methods used for multiple sensor rainfall estimation, 1, Experimental design, *Water Resour. Res.*, **26**, 469–478.

Smith, E.A., Mugnai, A., Cooper, H.J., Tripoli, G.J. and Xiang, X., 1992. Foundations for statistic-physical precipitation retrieval from passive microwave satellite measurements, Part I: Brightness temperature properties of a time-dependent cloud radiation model, *J. Appl. Meteorol.*, **31**, 506–531.

Smith, J.A. and Karr, A.F., 1985. Parameter estimation for a model of space-time rainfall, *Water Resour. Res.*, **21**, 1251–1258.

Smith, J.A. and Krajewski, W.F., 1987. Statistical modeling of rainfall using radar and raingauge observations, *Water Resour. Res.*, **23**, 1893–1900.

Tompson, A.F.B., Ababou, R. and Gelhar, L.W., 1989. Implementation of the three-dimensional turning band random field generator, *Water Resour. Res.*, **25**, 2227–2243.

Waymire, E., 1985. Scaling limits and self-similarity in precipitation fields, *Water Resour. Res.*, **21**, 1271–1281.

Waymire, E. and Gupta, V.K., 1981a. The mathematical structure of rainfall representation, 1, A review of the stochastic rainfall models, *Water Resour. Res.*, **17**, 1261–1272.

Waymire, E. and Gupta, V.K., 1981b. The mathematical structure of rainfall representation, 2, A review of the theory of point processes, *Water Resour. Res.*, **17**, 1273–1285.

Waymire, E. and Gupta, V.K., 1981c. The mathematical structure of rainfall representation, 3, Some application of the point process theory to rainfall processes, *Water Resour. Res.*, **17**, 1287–1294.

Waymire, E., Gupta, V.K. and Rodriguez-Iturbe, I., 1984. A spectral theory of rainfall intensity at the meso-beta scale, *Water Resour. Res.*, **20**, 1453–1465.

Wilson, C.B., Valdes, J.B. and Rodriguez-Iturbe, I., 1979. On the influence of the spatial distribution of rainfall on storm runoff, *Water Resour. Res.*, **15**, 321–328.

Zawadski, I., 1987. Fractal structure and exponential decorrelation in rain, *J. Geophys. Res.*, **92**, 9586–9590.